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ADJOINT VARIATIONAL METHODS IN NONCONSERVATIVE STABILITY PROBLEMS†

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Abstract—A general nonself-adjoint eigenvalue problem is examined and it is shown that the commonly employed approximate methods, such as the Galerkin procedure, the method of weighted residuals and the least square technique lack variational descriptions. When used in their previously known forms they do not yield stationary eigenvalues and eigenfunctions. With the help of an adjoint system, however, several analogous variational descriptions may be developed and it is shown in the present study that by properly restating the method of least squares, stationary eigenvalues may be obtained. Several properties of the adjoint eigenvalue problem, known only for a restricted group, are shown to exist for the more general class selected for study.

1. INTRODUCTION

THE investigation of the eigenvalues of a nonself-adjoint differential operator may be facilitated by the variational method if the concept of adjoint system is introduced. The idea was seemingly first suggested by Morse and Feshbach [1] and further explored successfully by Chandrasekhar [2] in his studies of hydrodynamic stability. In spite of this success in hydrodynamic and hydromagnetic stability problems, the idea does not seem to have attracted the attention of researchers in nonconservative stability problems of elastic solids. In a previous study [3] the authors have discussed how an adjoint system to the problem of an elastic continuum subjected to purely follower-type surface tractions may be constructed. It was shown that the two sets of eigenvalues of the original and the adjoint systems are identical and that each member of the set is a stationary value for a variation of the displacement functions.

In this study we suggest several formulations of the variational description of the eigenvalues of a complex differential equation in which the eigenvalues appear in the coefficients of the operator as analytic functions. In this form the eigenvalue problem is much more involved than the problems outlined by Morse and Feshbach [1], Chandrasekhar [2] and Roberts [4]. Their results are restricted to regular eigenvalue problems and are not applicable directly to the present case. The treatment of the present study is believed

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to yield all the main results, however, in modified forms, of the regular systems discussed by earlier authors [1, 2, 4, 5].†

The need for developing such approximate methods of solving more general nonconservative stability problems which, on one hand, would be based at least partially on a firm mathematical foundation, and, on the other hand, would provide effective means for numerical treatment, has been emphasized by Bolotin [6], Herrmann [7] and several other authors in aeroelasticity [8, 9]. In the past, the trend has been to employ the method of Galerkin which, unfortunately, does not provide an estimate of the order of magnitude of the error involved, nor does it, in general, guarantee convergence. The proof of convergence of the Galerkin method for nonself-adjoint boundary value problems has been given for only a few simple problems [10–13]. Past studies indicate that several attempts were made to formulate variational principles by other means for nonself-adjoint and nonlinear systems. These formulations, however, are found to lack the advantages of genuine variational principles, mainly because the integral is not stationary or because no variational integral exists. Examples may be cited in recent works of Glansdorff and Prigogine, known as the *method of local potential*, and in Biot's *Lagrangian thermodynamics*.

Another aspect of the present study is a comparison of several different approximate methods, namely, the Galerkin procedure and the method of weighted residuals, with the proposed variational method. It will be shown that these other methods become meaningful as special cases of the proposed method only under special circumstances and in general they do not possess the advantages of variational principles. Additionally, it turns out that with the help of adjoint systems, a reformulation of the method of least squares may be made, which yields stationary eigenvalues of the system. Particularly, this latter finding is believed to be of considerable interest and in a future study it is proposed to explore these various methods numerically.

2. STATEMENT OF PROBLEM

Consider the following form of an ordinary linear differential equation:

$$Pu = \omega Q u \tag{1}$$

where

$$\begin{aligned} P &= \sum_{n=0}^{N} \alpha_{N-n}(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} \\ Q &= \sum_{n=0}^{K} \beta_{K-n}(x,\omega)\frac{\mathrm{d}^n}{\mathrm{d}x^n}, \qquad K \le N. \end{aligned} \tag{2}$$

In the above, u denotes a function of a real variable x for $a \le x \le b$, and α_n and β_n are continuous functions of x whose $N-n$ derivatives with respect to x exist and are continuous. Further, α_0 does not vanish at any point of the closed interval (a, b).

Although the parameter ω may be regarded as the eigenvalue of the system, in stability analysis it usually denotes frequency of small oscillations. Equation (1) thus defines a

† When the present manuscript was completed, the authors' attention was drawn to the studies of Biot [26] and Flax [27] who also employed the concept of adjoint operator. Their studies, however, were restricted to the static problem of aeroelastic divergence.

wide range of one-dimensional, dynamical systems of autonomous elastic bodies whose motion follows the form $u(x)\,e^{i\omega t}$. The operator Q is a complex-valued analytic function of ω and its origin is due to velocity-dependent forces in the system. Serious doubts have been raised in the past regarding the validity of solutions of that group of problems in which the forces are idealized to be of purely follower type. The role played by velocity-dependent forces in influencing stability of equilibrium of nonconservative elastic systems has been recognized to be especially intriguing [6, 7]. Consequently, in many problems the analysis will have to include terms originating from Coriolis acceleration or from other gyroscopic effects, viscous damping, etc. The operator Q, therefore, encompasses the effect of such forces and the definition (1) is believed to be fairly general. Note that in the absence of velocity-dependent forces $Q \equiv 1$.

Associated with (1) we consider N linear, homogeneous, boundary conditions in $u(a), u'(a), \ldots, u^{(N-1)}(a), u(b), u'(b), \ldots, u^{(N-1)}(b)$ as given by

$$L_j u = 0, \qquad j = 1, 2, \ldots N \tag{3}$$

where

$$L_j = \sum_{n=0}^{N-1} \{\eta_{jn} + \omega\theta_{jn}\} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \tag{4}$$

η_{jn} and θ_{jn} are quantities characterizing certain properties (such as stiffness or inertia) at the end points (a, b). θ_{jn} may, further, be assumed to be continuous, single-valued functions of ω. For future use let us define N additional forms $L_{N+1}u, \ldots L_{2N}u$ in $u^i(a)$, $u^i(b)$ so that $L_1 u, L_2 u, \ldots, L_{2N}u$ are linearly independent.

From (1) we obtain, after integrating by parts,

$$\int_a^b \{u^*[Pu - \omega Qu] - u[P^*u^* - \omega Q^*u^*]\}\,\mathrm{d}x = [P_1(u, u^*)]_a^b \tag{5}$$

where

$$P^*u^* = (-1)^N \frac{\mathrm{d}^N(\alpha_0 u^*)}{\mathrm{d}x^N} + (-1)^{N-1} \frac{\mathrm{d}^{N-1}(\alpha_1 u^*)}{\mathrm{d}x^{N-1}} + \cdots - \frac{\mathrm{d}(\alpha_{N-1}u^*)}{\mathrm{d}x} + \alpha_N u^* \tag{6}$$

$$Q^*u^* = (-1)^K \frac{\mathrm{d}^K(\beta_0 u^*)}{\mathrm{d}x^K} + (-1)^{K-1} \frac{\mathrm{d}^{K-1}(\beta_1 u^*)}{\mathrm{d}x^{K-1}} + \cdots - \frac{\mathrm{d}(\beta_{K-1}u^*)}{\mathrm{d}x} + \beta_K u^*, \qquad K \leq N \tag{7}$$

and $[P_1(u, u^*)]_a^b$ is a bilinear form in $u(a), u'(a), \ldots, u^{(N-1)}(a), u(b), \ldots, u^{(N-1)}(b)$ and $u^*(a), u^{*\prime}(a), \ldots, u^{*(N-1)}(a), u^*(b), \ldots, u^{*(N-1)}(b)$. We may write

$$[P_1(u, u^*)]_a^b = \sum_{n=1}^{2N} L_n u L^*_{2N+1-n} u^* \tag{8}$$

in which $L_n^* u^*$ are linear in $u^*(a), u^{*\prime}(a), \ldots, u^{*(N-1)}(a), u^*(b), \ldots, u^{*(N-1)}(b)$. By virtue of (3), N terms of the right-hand side of (8) will be zero and therefore the remaining N terms will also be zero if we select N adjoint boundary conditions of the type

$$L_j^* u^* = 0, \qquad j = 1, 2, \ldots, N \tag{9}$$

where L_j^* may be written in the following general form:

$$L_j^* = \sum_{n=0}^{N-1} \{\eta_{jn}^* + \omega\theta_{jn}^*\} \frac{\mathrm{d}^n}{\mathrm{d}x^n}. \tag{10}$$

Note that the parameters η_{jn}^*, θ_{jn}^* will depend upon the choice of the additional N linearly independent forms of $L_{N+1}, L_{N+2}, \ldots, L_{2N}$. Since these N additional conditions may be arbitrarily selected, with the only restriction of being linearly independent, the adjoint boundary conditions are not unique. Therefore, the adjoint boundary value problem is not uniquely defined. By way of illustration, this fact was apparently brought into attention for the first time by Roberts [4]. Of course, η_{jn}^* and θ_{jn}^* will be functions of η_{jn}, θ_{jn}, α_n and β_n of the original problem.

Thus, we have obtained the following system as being an adjoint to the eigenvalue problem defined by (1) and (3):

$$P^*u^* = \omega^* Q^* u^* \tag{11}$$

with the boundary conditions

$$L_j^* u^* \equiv \sum_{n=0}^{N-1} \{\eta_{jn}^* + \omega^* \theta_{jn}^*\} \frac{d^n u^*}{dx^n} = 0, \qquad j = 1, 2, \ldots, N. \tag{12}$$

We note the following property of u, u^*:

$$\int_a^b u^*[Pu - \omega Q u]\,dx = \int_a^b u[P^*u^* - \omega Q^* u^*]\,dx. \tag{13}$$

3. PROPERTIES OF ADJOINT SYSTEMS

Let $\{\omega_i\}$ and $\{u_i\}$ denote, respectively, the sets of eigenfrequencies and eigenmodes of the original and $\{\omega_i^*\}$ and $\{u_i^*\}$ those of adjoint systems. We will suppose that both sets of eigenmodes $\{u_i\}$ and $\{u_i^*\}$ span the domain defined by the independent variable x.

The development of a variational principle for system (1)–(4) begins with the proof of the two sets of eigenvalues $\{\omega_i\}$ and $\{\omega_i^*\}$ being identical. We supply the proof in the following manner. Let us select an arbitrary eigenvalue ω_i of the original problem whose corresponding eigenfunction is u_i. Since the adjoint system is governed by an Nth order differential equation and, therefore, possesses N linearly independent solutions, we can find for this value of ω a function u_i^* which satisfies any set of $N-1$ members of N boundary conditions (9). One then shows that the function u_i^* must necessarily also satisfy the remaining Nth boundary condition and hence is an eigenfunction of the adjoint system whose eigenvalue is $\omega_i^* = \omega_i$. The crux of the problem lies in proving that if the Nth boundary conditions is not satisfied it gives rise to a contradiction. As we will learn later, this contradiction is a statement that the set of eigenfunctions $\{u_i\}$ is a trivial set, i.e. $\{u_i\} \equiv 0$.

Let us assume that we have constructed a solution u_i^* of (11) with $\omega^* = \omega_i$, which satisfies only $N-1$ boundary conditions

$$L_j^* u_i^* = 0, \qquad j = 1, 2, \ldots, N; j \neq l. \tag{14}$$

We now prove that $L_l u_i^*$ must also be zero. To do this, multiply (1) and (11) by u_i^* and u_i, respectively, and integrate over the interval (a, b). Then we subtract one of the resulting equations from the other to yield

$$\int_a^b \{u_i^*(P - \omega_i Q)u_i - u_i(P^* - \omega_i Q^*)u_i^*\}\,dx = \sum_{n=1}^{2N} L_n u_i L_{2N+1-n}^* u_i^* = 0 \tag{15}$$

by virtue of (13). If we use, now, the boundary conditions of u_i and u_i^*, we have

$$L_{N+l}u_i L_l^* u_i^* = 0. \tag{16}$$

Thus, if $L_{N+l}u_i \neq 0$, then we have $L_l^* u_i^* = 0$, so that ω_i is an eigenvalue of the adjoint system. On the other hand, if $L_{N+l}u_i = 0$, the number of linearly independent boundary conditions to be satisfied by the original system would be arbitrarily larger than N and, therefore, the set of eigenfunctions $\{u_i\}$ would be trivial. Consequently, $\{\omega_i\}$ and $\{\omega_i^*\}$ are identical sets of eigenvalues. For illustration, consider the following eigenvalue problem:

$$\frac{d^4u}{dx^4}+F^2\frac{d^2u}{dx^2}+2i\beta F\omega\frac{du}{dx}-\omega^2u = 0, \qquad 0 < x < 1 \tag{17}$$

$$\begin{aligned} u = \frac{du}{dx} = 0 \quad \text{at } x = 0 \\ \frac{d^2u}{dx^2} = \frac{d^3u}{dx^3} = 0 \quad \text{at } x = 1. \end{aligned} \tag{18}$$

System (17), (18) is nonself-adjoint and governs over-all motions of a cantilevered elastic pipe conveying an incompressible fluid at a constant velocity [14]. An adjoint system to (17), (18) may be described by the following set of equations:

$$\frac{d^4u^*}{dx^4}+F^2\frac{d^2u^*}{dx^2}-2i\beta F\omega^*\frac{du^*}{dx}-\omega^{*2}u^* = 0 \tag{19}$$

$$\begin{aligned} u^* = \frac{du^*}{dx} = 0 \quad \text{at } x = 0 \\ \frac{d^2u^*}{dx^2}+F^2u^* = 0 \quad \text{at } x = 1 \\ \frac{d^3u^*}{dx^3}+F^2\frac{du^*}{dx}-2i\beta F\omega^*u^* = 0 \quad \text{at } x = 1. \end{aligned} \tag{20}$$

Following the procedure outlined previously, we suppose to have constructed a solution u_i^* of (19) with $\omega^* = \omega_i$ which satisfies only the following conditions:

$$u_i^* = \frac{du_i^*}{dx} = 0 \quad \text{at } x = 0$$

$$\frac{d^3u_i^*}{dx^3}+F^2\frac{du_i^*}{dx}-2i\beta F\omega_i u_i^* = 0 \quad \text{at } x = 1$$

and

$$\frac{d^2u_i^*}{dx^2}+F^2u_i^* \neq 0 \quad \text{at } x = 1.$$

We now multiply (17) by u_i^* and (19) by u_i and integrate over (0, 1) to obtain finally

$$\left(\frac{d^2u_i^*}{dx^2}+F^2\frac{du_i^*}{dx}-2i\beta F\omega_i u_i^*\right)u_i = 0 \quad \text{at } x = 1. \tag{21}$$

Thus if the expression in parentheses in (21) is not zero, then $u_i = 0$ at $x = 1$ and this along with the other four boundary conditions as given by (18) would imply that the system has only a trivial solution, i.e. $\{u_i\} \equiv 0$. Thus we must have

$$\frac{d^2u_i^*}{dx^2}+F^2\frac{du_i^*}{dx}-2i\beta F\omega_i u_i^* = 0 \quad \text{at } x = 1$$

and, therefore, the sets $\{\omega_i\}$ and $\{\omega_i^*\}$ are identical.

To investigate orthogonality let us consider

$$Pu_i = \omega_i Qu_i \tag{22a}$$

and

$$P^*u_j^* = \omega_j Q^*u_j^*. \tag{22b}$$

Multiplying (22a) and (22b) by u_j^* and u_i, respectively, and integrating over (a, b), the following is obtained if we take a difference of the resulting equations:

$$\int_a^b \{u_j^*(Pu_i-\omega_i Qu_i)-u_i(P^*u_j^*-\omega_j Q^*u_j^*)\}\, dx = 0$$

which reduces to

$$(\omega_j-\omega_i)\int_a^b u_i Q^*u_j^*\, dx = 0; \qquad (\omega_j-\omega_i)\int_a^b u_j^* Qu_i\, dx = 0.$$

Thus, if $\omega_i \neq \omega_j$, we have

$$\int_a^b u_i Q^*u_j^*\, dx = 0; \qquad \int_a^b u_j^* Qu_i\, dx = 0 \tag{23}$$

which are the modified bi-orthogonality relationships.

The expression given by (23) is analogous to the orthogonality of the eigenfunctions of self-adjoint problems. For the investigation of several initial value problems in self-adjoint systems it is expedient to normalize the eigenfunctions. It turns out that a general normalization for the present case does not follow. The results of certain special cases, however, are known. Thus when $Q = 1$, Ince [15] has shown that

$$\int_a^b u_i u_i^* \neq 0.$$

Further, it may be asserted that when Q is independent of ω, normalization may be achieved by

$$\int_a^b u_i Q^*u_i^* = 1. \tag{24}$$

Let us now investigate the extremum property of the eigenvalues ω_i. We write

$$\omega_i \int_a^b u_i^* Qu_i\, dx = \int_a^b u_i^* Pu_i\, dx. \tag{25}$$

Taking the variation of (25) in such a way that δu, and δu^*, satisfy the boundary conditions (3) and (12), respectively, we obtain

$$\delta\omega_i \int_a^b u_i^* Q u_i \,\mathrm{d}x + \omega_i \int_a^b \left\{\delta u_i^* Q u_i + \delta\omega_i u_i^* \frac{\partial Q}{\partial \omega_i} u_i + u_i^* Q \delta u_i\right\} \mathrm{d}x = \int_a^b (\delta u_i^* P u_i + u_i^* P \delta u_i)\,\mathrm{d}x$$

which may be rearranged to yield

$$\delta\omega_i = \frac{\int_a^b \{\delta u_i^*(P u_i - \omega_i Q u_i) + u_i^*(P\delta u_i - \omega_i Q \delta u_i)\}\,\mathrm{d}x}{\int_a^b u_i^*[Q u_i + (\partial Q/\partial\omega_i) u_i]\,\mathrm{d}x}. \tag{26}$$

The second term within braces in the numerator of (26) may be transformed, using (13), to yield the following:

$$\delta\omega_i = \frac{\int_a^b \delta u_i^*(P u_i - \omega_i Q u_i)\,\mathrm{d}x + \int_a^b \delta u_i (P^* u_i^* - \omega_i Q^* u_i^*)\,\mathrm{d}x}{\int_a^b u_i^*[Q u_i + (\partial Q/\partial\omega_i) u_i]\,\mathrm{d}x}. \tag{27}$$

Hence, if equations (1) and (11) are obeyed, $\delta\omega_i$ is zero to first order for all small, arbitrary variations in δu_i and δu_i^* satisfying (3) and (12), respectively. Obviously, the converse is also true. Thus a definite statement may be made regarding the error involved in stipulating that the eigenvalues are stationary values.

During this search for obtaining expressions which yield stationary eigenvalues, it was discovered that another functional of u_i and u_i^* also exists which has similar properties. Approximate solutions based upon this functional parallel the method of least squares in self-adjoint boundary value problems and, therefore, a restatement of the least squares technique for nonself-adjoint systems is formulated in a later section on other approximate methods.

4. AN APPROXIMATE METHOD OF STABILITY ANALYSIS

The extremum property of the eigenvalues ω_i, as expressed by (27), suggests an approximate procedure for their determination, in the spirit of approximate methods for self-adjoint systems based on variational principles. We may select two sets of trial functions $u_i(a_1, a_2, \ldots)$ and $u_i^*(a_1^*, a_2^*, \ldots)$ which satisfy the appropriate boundary conditions and contain undetermined parameters a_j and a_j^*. An approximate expression of the eigenvalues ω is obtained, by using equation (25), as a function of these parameters. A stationary value of ω is then obtained by determining the parameters from equations of the type

$$\frac{\partial\omega}{\partial a_j} = 0; \qquad \frac{\partial\omega}{\partial a_j^*} = 0. \tag{28}$$

Considering the general nature of the problem under investigation, it is difficult to say what would be the best choice in the selection of trial functions. Certainly any information, such as symmetry, or prior experience with related problems should be exploited, but there seems to be no way available at present to do this systematically. Usually, however, several sets of approximating functions may be available and a selection may be made either on the basis of ease of integrations or based upon rapid convergence of the problem. Note that for good approximations in most situations, the sets of trial functions should be complete in the domain of the problem. In hydrodynamic and hydromagnetic stability problems Chandrasekhar [2] employed complete sets of eigenfunctions by solving one or another lower-order, simpler but related eigenvalue problem on the same domain. Lee and Reynolds [16] have discussed the use of orthogonal polynomials in the stability of parallel flows. Since, in all these problems the original and the adjoint boundary conditions were identical, the analysis was simplified. A further discussion of this aspect will follow.

Let us assume that we have selected two sets of linearly independent trial functions $v_i(x)$ and $v_i^*(x)$, $i = 1, 2, \ldots \infty$, both of which span the domain of the system. These sets satisfy their respective boundary conditions (3) and (12) and thus they may be identical only if (3) and (12) represent the same boundary conditions. With the help of unknown coefficients a_i and a_i^*, the following approximations are assumed:

$$u(x) = \sum_{i=1}^{\infty} a_i v_i(x) \tag{29a}$$

$$u^*(x) = \sum_{i=1}^{\infty} a_i^* v^*(x). \tag{29b}$$

Now, we construct an infinite-dimensional space in $a_1, a_2, \ldots, a_1^*, a_2^*, \ldots$, in such a way that the eigenvalue ω is determined as an extremum value in this space. For this purpose consider the following integral

$$\omega \int_a^b u^* Q u \, \mathrm{d}x = \int_a^b u^* P u \, \mathrm{d}x$$

which, after substitution of (29), yields

$$\int_a^b \left[\sum_{m,n=1}^{\infty} (Pv_m - \omega Q v_m) v_n^* a_m a_n^* \right] \mathrm{d}x = 0. \tag{30}$$

It is well to emphasize at this point that the coefficients of the operator Q are functions of ω. Treating (30) as an implicit function of ω in a_m and a_n^*, we obtain the following derivatives with respect to a_m and a_n^*:

$$\int_a^b \left[\sum_{n=1}^{\infty} (Pv_m - \omega Q v_m) v_n^* a_n^* - \sum_{m,n=1}^{\infty} \frac{\partial \omega}{\partial a_m} \left(Q v_m + \frac{\partial Q}{\partial \omega} v_m \right) v_n^* a_m a_n^* \right] \mathrm{d}x = 0 \tag{31}$$

and

$$\int_a^b \left[\sum_{m=1}^{\infty} (Pv_m - \omega Q v_m) v_n^* a_m - \sum_{m,n=1}^{\infty} \frac{\partial \omega}{\partial a_n^*} \left(Q v_m + \frac{\partial Q}{\partial \omega} v_m \right) v_n a_m a_n^* \right] \mathrm{d}x = 0. \tag{32}$$

In order to obtain a stationary value of ω we must set $\partial\omega/\partial a_m = \partial\omega/\partial a_n^* = 0$ in (31)–(32) to obtain

$$\sum_{n=1}^{\infty} a_n^* \int_a^b (Pv_m - \omega Q v_m) v_n^* \, \mathrm{d}x = 0 \tag{33}$$

and

$$\sum_{m=1}^{\infty} a_m \int_a^b (Pv_m - \omega Q v_m) v_n^* \, \mathrm{d}x = 0. \tag{34}$$

Equations (33) and (34) are two linear homogeneous matrix equations in a_m^* and a_n, which are the transpose of each other. Therefore, their eigenvalue ω will be the same. In the sequel we will consider only (34) which yields the following secular equation:

$$\det \Delta = |A_{mn} - \omega B_{mn}| = 0 \tag{35}$$

where

$$\begin{aligned} A_{mn} &= \int_a^b P v_m v_n^* \,\mathrm{d}x \\ B_{mn} &= \int_a^b Q v_m v_n^* \,\mathrm{d}x. \end{aligned} \tag{36}$$

The determinant Δ is of infinite order, but in practice usually a good approximation may be achieved by retaining a finite number of terms.

5. OTHER APPROXIMATE METHODS

We now wish to analyze the relationships between the adjoint variational method proposed in the present study and several other methods, in particular the method of weighted residuals, the Galerkin procedure and the method of least squares. Several mathematical aspects of these methods, such as convergence and estimation of error, are known in the case of self-adjoint problems [17, 18], and their application to a general problem, not necessarily linear and self-adjoint, is justified only heuristically.

Examining the general aspect of nonconservative systems in which the processes involved, such as dissipation and exchange of energy between restoring and applied forces, are irreversible, it is appropriate to mention briefly some recent developments in heat conduction and other transport phenomena [19–21]. Based upon the concept of local potential, Glansdorff and Prigogine [19, 22, 23] have developed a restricted variational principle. Certain applications and generalization of this principle beyond heat conduction and transport phenomena have been made; however, its physical significance is not sufficiently clear to investigate the problem of elastic stability of nonconservative systems. Roberts [24] has shown that the technique of Glansdorff and Prigogine is equivalent to a particular form of Galerkin's method when applied to certain steady-state situations. Biot's treatment of Lagrangian thermodynamics [20, 21] centers on certain variational formulation. His technique permits approximate solutions of problems involving non-linearities. This method, however, has not been applied to investigate solutions of systems under discussion.

In the method of weighted residuals, one stipulates that the trial solution satisfies the differential equation (1) in some definite sense (see a review article by Finlayson and Scriven [25]). This notion is made more specific by requiring that the weighted integrals of the residuals are set equal to zero:

$$\langle f_j, (Pg - \omega Qg)\rangle = 0; \qquad j = 1, 2, \ldots, N \tag{37}$$

where

$$\langle f, g\rangle = \int_a^b fg \,\mathrm{d}x$$

represents a spatial average or inner product, f_j is a prescribed weighting function and g is a trial solution. The selection of trial functions g_j remains somewhat dependent on the user's intuition and experience. The criteria discussed earlier, however, are valid even in this case and there is no unified approach to make the best choice which always gives accurate and rapidly convergent results. The trial solution g is expressed as

$$g(x) = \sum_{i=1}^{N} a_i g_i(x)$$

a substitution of which in (37) results in

$$\sum_{i=1}^{N} a_i\{\langle f_j, Pg_i\rangle - \omega\langle f_j, Qg_i\rangle\} \equiv \sum_{i=1}^{N} a_i(A_{ji} - \omega B_{ji}) = 0. \tag{38}$$

Thus, by comparing (38) with (33) we find that the form by which ω is determined is identical.

In the past, not much was known regarding the selection of the weighting functions f_j, which yields an estimation of the error involved. In view of the present development it may now be emphasized that a proper choice of the weighting functions is the one that satisfies the adjoint boundary conditions (12) and, therefore, yields stationary values in the space of trial functions for infinitesimal variations. Because Galerkin's method is a special case of the method of weighted residuals ($f_j \equiv g_j$), it suffers from the same limitations. However, if the adjoint boundary conditions (12) coincide identically with (3), the proposed adjoint variational and Galerkin methods are formally similar.

6. REFORMULATION OF THE METHOD OF LEAST SQUARES

The method of least squares is also a possible tool available for approximate calculations of the eigenvalues. Although this particular method has not been widely employed in stability problems, it seems desirable to investigate certain aspects of this method by means of the present analysis. In order to understand clearly the method of least squares as applied to nonself-adjoint eigenvalue problems, let us consider the regular problem by assuming $Q \equiv 1$.

Mikhlin [17, chapter X] has discussed certain mathematical aspects of the method of least squares as applied to self-adjoint boundary value problems. For this group of problems the method reduces to the solution of the following eigenvalue problem:

$$\langle (R-\omega)g, Rg\rangle \equiv \sum_{i=1}^{N} a_i\{\langle Rg_i, Rg_j\rangle - \omega\langle g_i, Rg_j\rangle\} \equiv \sum_{i=1}^{N} a_i(A_{ij} - \omega B_{ij}) = 0 \tag{39}$$

where R is a self-adjoint operator and g_j are a set of trial functions which satisfy the boundary conditions associated with R. Mikhlin has shown that if $\{g_j\}$ is a complete set, the solution converges to the exact one. In this form the application of this method to nonself-adjoint systems seems dubious and the purpose now is to investigate the usefulness of the following reformulation of the method of least squares. We consider the following integral:

$$\omega_i\langle u_i, P^*u_i^*\rangle = \langle Pu_i, P^*u_i^*\rangle \tag{40}$$

where P^* is an adjoint operator to P as defined in (5), u_i and u_i^* are eigenfunctions of the original and the adjoint systems corresponding to the same eigenvalue ω_i.

Taking the variation of (40) in such a way that δu_i and δu_i^* satisfy the respective boundary conditions, we find

$$\delta\omega_i = \frac{\langle P^*\delta u_i^*, (Pu_i - \omega_i u_i)\rangle + \langle P^*u_i^*, (P\delta u_i - \omega_i \delta u_i)\rangle}{\langle u_i, P^*u_i^*\rangle}. \tag{41}$$

By using the property similar to (13), the second integral in the numerator of (41) is transformed to yield finally

$$\delta\omega_i = \frac{\langle P^*\delta u_i^*, (Pu_i - \omega_i u_i)\rangle + \langle (P\delta u_i, (P^*u_i^* - \omega_i u_i^*)\rangle}{\langle u_i, P^*u_i^*\rangle}. \tag{42}$$

Thus, if the field equations of u_i and u_i^* are obeyed, we find from (42) that $\delta\omega_i$ is zero to first order for all small, arbitrary variations in δu_i and δu_i^* satisfying the corresponding boundary conditions. Again, the converse is also true.

Thus we find that the integral expression (40) has properties similar to (25) and consequently a definite statement may be made in stipulating that eigenvalues obtained from (40) are stationary values. Based upon the discussions of the previous sections, we may select trial functions $v_i(x)$ and $v_i^*(x)$ and obtain, similarly, from (40) the following set of linear homogeneous matrix equations in unknown parameters a_i and a_i^*,

$$\sum_{i=1}^{N} a_j^*\{\langle Pv_i, P^*v_j^*\rangle - \omega\langle v_i, P^*v_j\rangle\} \equiv \sum_{i=1}^{N} a_j^*(A_{ij} - \omega B_{ij}) = 0 \tag{43}$$

and

$$\sum_{i=1}^{N} a_i\{\langle Pv_i, P^*v_j^*\rangle - \omega\langle v_i, P^*v_j\rangle\} = \sum_{i=1}^{N} a_i(A_{ji} - \omega B_{ji}) = 0 \tag{44}$$

from which approximate calculations may be made for $\{\omega_i\}$. We notice that equations (39) and (43) are similar in form and, therefore, may suggest that (40) is a restatement of the principle of least squares for nonself-adjoint eigenvalue problem. If the system is self-adjoint, then $P = P^*$ and $u = u^*$ and (40) reduces to the form as described by Mikhlin.

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Абстракт—Исследуется несамосопряженная задача на собственные значения и указывается, что обычно использованные приближенные методы, в роде способа галеркина, метода весовых вычётов и метода наименьших квадратов, испытывают недостаток вариационного формализма. Когда они применяются в своих предидущих формах, тогда они не дают стационарных собственных значений и собственных функций. Однако, с помощью сопряженной системы, можно определить некоторый аналогичный вариационный формализм. В настоящей работе указывается, что применяя как следует, вновь, метод наименьших квадратов, можно получить стационарные собственные значения. Показано, что для более широкого класса, выбранного для исследований, существуют некоторые свойства сопряженной проблемы на собственные значения, известные только для ограниченной группы.